Analysis II: Basic knowledge of real analysis: Part I

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Sets

What is the term "Set"?: A set is a collection of objects.

For example,

$\{a, b, c\}$ is a set which has 3 elements.

$\mathbb{N} = \{x | \ x \text{ is a natural number}\} = \{1, 2, 3, 4, \cdots\}$

$\mathbb{Z} = \{x | \ x \text{ is an integer}\}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

$\mathbb{Q} = \{x | \ x \text{ is a rational number}\}$

$\mathbb{R} = \{x | \ x \text{ is a real number}\}$

The number of elements of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: Infinity!

We say that a set with a finite number of elements is finite, and a set with an infinite number of elements is infinite.

The empty set: the set with no elements and is denoted by $\emptyset$.

Countable and uncountable sets

Definition

A countable set: a set with the same cardinality (i.e., number of elements) as some subset of the set of natural numbers.
An uncountable set: a set that is not countable.

The term was originated by Georg Cantor (1845–1918, Germany).
The elements of a countable set can be counted one at a time – although the counting may never finish, every element of the set will eventually be associated with a natural number.
More rigorously,

Definition

A set S is called countable if there exists an injective (one-to-one) function f from S to the natural numbers $\mathbb{N}$.

So

$\mathbb{N}$ is countable.

$\mathbb{Z}$ is countable.

$\mathbb{Q}$ is countable.

But $\mathbb{R}$ is uncountable!!!

Definition

Two sets X and Y have the same cardinality if there exists a one-to-one correspondence (bijective) from X to Y.

When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$.
(write $|S| = \aleph_0$. $\aleph_0$ is called "aleph null" or "aleph naught".)
So
$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$. $|\mathbb{R}| = \mathfrak{c} = \aleph$.

Least upper bound

Definition

Let X be a nonempty subset of R. A number a in $\mathbb{R}$ is said to be a least upper bound for X if

(i) a is an upper bound for X. ($x \leq a$ for all $x \in X$.)

(ii) If b is an upper bound for X, then $a \leq b$.

We write $\mathbf{lub}\, X = a$ or $\mathbf{sup}\, X = a$. The least upper bound of a set X is also called the supremum of X.

The least upper bound (LUB) axiom

If a nonempty set of real numbers has an upper bound, then it has a least upper bound.

Greatest lower bound

Definition

Let $\boldsymbol{X}$ be a nonempty subset of $\boldsymbol{R}$. A number $\boldsymbol{a}$ in $\mathbb{R}$ is said to be a greatest lower bound for $\boldsymbol{X}$ if

(i) $\boldsymbol{a}$ is a lower bound for $\boldsymbol{X}$. ($\boldsymbol{x} \geq \boldsymbol{a}$ for all $\boldsymbol{x} \in \boldsymbol{X}$.)

(ii) If $\boldsymbol{b}$ is a lower bound for $\boldsymbol{X}$, then $\boldsymbol{a} \geq \boldsymbol{b}$.

We write $\mathbf{glb}\, \boldsymbol{X} = \boldsymbol{a}$ or $\mathbf{inf}\, \boldsymbol{X} = \boldsymbol{a}$. The greatest lower bound of a set $\boldsymbol{X}$ is also called the infimum of $\boldsymbol{X}$.

The greatest lower bound (GLB) axiom

If a nonempty set of real numbers has a lower bound, then it has a greatest lower bound.

Limit of a function

Definition

We say that the limit of a function $f(x)$ as x approaches a is L if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. We write $\lim_{x \to a} f(x) = L$.

(Example) Consider $f(x) = 3x - 1$.
(In Calculus I) $\lim_{x \to 1} f(x) = \lim_{x \to 1} (3x - 1) = 2$.

In Analysis, we must prove this rigorously using the above definition!

We can show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

Proof

Show $\lim_{x \to 1}(3x - 1) = 2$.

$$\Downarrow$$

Find δ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

(Proof)

Let $\epsilon > 0$.

Suppose $0 < |x - 1| < \delta$.

Then $|(3x - 1) - 2| = |3(x - 1)| = 3|x - 1| < 3\delta$.

If we set $\delta = \frac{\epsilon}{3}$, $|(3x - 1) - 2| < \epsilon$ is satisfied.

Limit of a sequence

Definition

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ $(a_n \in \mathbb{R})$ has a limit $L \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a positive integer N such that $|a_n - L| < \epsilon$ whenever $n \geq N$. We write $\lim_{n\to\infty} a_n = L$. Sequences that have limits are called convergent. Sequences that do not have limits are called divergent.

(Example) Consider $a_n = \frac{n}{n+1}$.

(In Calculus I) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$

In Analysis, we must prove this rigorously using the above definition!

Proof

Show $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$.

$\Downarrow$

Find N such that $|\frac{n}{n+1} - 1| < \epsilon$ whenever $n \geq N$.

(Proof)

Let $\epsilon > 0$.

Suppose $n \geq N > 0$.

Then $|\frac{n}{n+1} - 1| = |\frac{1}{n+1}| = \frac{1}{n+1} < \frac{1}{N+1}$.

If we set $\frac{1}{N+1} < \epsilon$ (i.e., $N > \frac{1}{\epsilon} - 1$), $|\frac{n}{n+1} - 1| < \epsilon$ is satisfied.