Analysis II: Basic knowledge of real analysis: Part II

Kenichi Maruno

Department of Mathematics, The University of Texas - Pan American

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Definition

Let x be a real number. We define

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

We call $|x|$ the absolute value of x.

1. Let $\epsilon > 0$. Then $|x| < \epsilon$ if and only if $-\epsilon < x < \epsilon$ and $|x| \leq \epsilon$ if and only if $-\epsilon \leq x \leq \epsilon$.
2. $x \leq |x|$ for all $x \in \mathbb{R}$.
3. $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.
4. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$. (the triangle inequality)

Sequence

Sequence

Let X be a set. A sequence of elements of X is a function from the set of positive integers into X.

Subsequence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let f be a strictly increasing function from $\mathbb{N}$ into $\mathbb{N}$. The sequence $\{a_{f(n)}\}_{n=1}^{\infty}$ is called a subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$. (The function f is strictly increasing if $f(m) < f(n)$ whenever $m < n$.)

Example: $a_n = \frac{1}{n}$, $\qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \dots$

$\Downarrow$

$a_{2n-1} = \frac{1}{2n-1}$, $\qquad 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$

Definition

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ ($a_n \in \mathbb{R}$) has a limit $L \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a positive integer N such that $|a_n - L| < \epsilon$ whenever $n \geq N$. We write $\lim_{n\to\infty} a_n = L$. Sequences that have limits are called convergent. Sequences that do not have limits are called divergent.

(Example) Consider $a_n = \frac{n}{n+1}$.

(In Calculus I) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{n+1} = \lim_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1.$

In Analysis, we must prove this rigorously using the above definition!

Proof

Show $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

$\Downarrow$

Find N such that $|\frac{n}{n+1} - 1| < \epsilon$ whenever $n \geq N$.

(Proof)

Let $\epsilon > 0$.

Suppose $n \geq N > 0$.

Then $|\frac{n}{n+1} - 1| = |\frac{1}{n+1}| = \frac{1}{n+1} \leq \frac{1}{N+1}$.

If we set $\frac{1}{N+1} < \epsilon$ (i.e., $N > \frac{1}{\epsilon} - 1$), $|\frac{n}{n+1} - 1| < \epsilon$ is satisfied.

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with limit L. Then any subsequence of $\{a_n\}_{n=1}^{\infty}$ has limit L.

(Proof)

Let $\{a_{f(n)}\}_{n=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. Let $\epsilon > 0$. There exists a positive integer N such that if $n \geq N$, then

$$|a_n - L| < \epsilon\,.$$

Let f be a strictly increasing function from $\mathbb{N}$ into $\mathbb{N}$. If $n \geq N$, then $f(n) \geq f(N) \geq N$ (if $f(N) \leq N$, f is not a strictly increasing function!), and therefore

$$|a_{f(n)} - L| < \epsilon\,.$$

Theorem

Let $\{a_n\}$ and $\{b_n\}$ be sequence such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then

$$\lim_{n\to\infty}(a_n + b_n) = L + M$$

(Proof)

Let $\epsilon > 0$. There exists a positive integer N_1 such that if $n \geq N_1$, then

$$|a_n - L| < \frac{\epsilon}{2}.$$

There exists a positive integer N_2 such that if $n \geq N_2$, then

$$|b_n - M| < \frac{\epsilon}{2}.$$

If $n \geq \max\{N_1, N_2\}$, then

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Bounded Sequences

Definition

We say that a sequence $\{a_n\}$ is bounded above (below) if there exists a number M such that $a_n \leq M$ $(a_n \geq M)$ for every positive integer n. We say that $\{a_n\}$ is bounded if $\{a_n\}$ is bounded both above and below for every positive integer n.

A sequence $\{a_n\}$ is bounded if and only if there exists a positive number M such that $|a_n| \leq M$ for every positive integer n.

Theorem

If $\{a_n\}$ is a convergent sequence, then $\{a_n\}$ is bounded.

(Proof)

Suppose $\lim_{n\to\infty} a_n = L$. If $\epsilon = 1$, there exists a positive integer N such that if $n \geq N$, then $|a_n - L| < 1$. Thus, if $n \geq N$, we have

$$|a_n| \leq |a_n - L| + |L| < 1 + |L|\,.$$

If $n < N$, then

$$|a_n| \leq \max\{|a_1|, |a_2|, ..., |a_{N-1}|\}\,.$$

Thus for any positive integer n,

$$|a_n| \leq M$$

if we let

$$M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |L|\}\,.$$

If $\{a_n\}$ is not bounded, then $\{a_n\}$ is a divergent sequence.

$\Downarrow$

the sequence $\{n\}_{n=1}^{\infty}$: not bounded $\Rightarrow$ divergent.

Remark:

$\{(-1)^n\}_{n=1}^{\infty}$: bounded and divergent.

Squeeze Theorem

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that

$$a_n \leq b_n \leq c_n$$

for every positive integer n. If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n \,,$$

then

$$\lim_{n\to\infty} b_n = L \,.$$

(Proof of Squeeze Theorem)

Let $\epsilon > 0$. There exists a positive integer N_1 such that if $n \geq N_1$, then

$$L - \epsilon < a_n < L + \epsilon\,.$$

There exists a positive integer N_2 such that if $n \geq N_2$, then

$$L - \epsilon < c_n < L + \epsilon\,.$$

Thus if $n \geq \max\{N_1, N_2\}$, we have

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon\,.$$

Definition

Let $\{a_n\}$ be a sequence. We say that $\{a_n\}$ diverges to infinity (minus infinity) and write $\lim_{n\to\infty} a_n = \infty$ ($\lim_{n\to\infty} a_n = -\infty$) if for every real number M, there exists a positive integer N such that if $n \geq N$, then $a_n > M (a_n < M)$.

Squeeze Theorem

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $a_n \leq b_n$ for every positive integer n. If $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.

Definition

Let $\{a_n\}$ be a sequence. We say that $\{a_n\}$ is increasing (decreasing) if $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$) for every positive integer n. We say that the sequence $\{a_n\}$ is monotone if either $\{a_n\}$ is increasing or $\{a_n\}$ is decreasing. If $a_n < a_{n+1}$ ($a_n > a_{n+1}$) for every positive integer n, we say that the sequence $\{a_n\}$ is strictly increasing (decreasing). We say that the sequence $\{a_n\}$ is strictly monotone if either $\{a_n\}$ is strictly increasing or strictly decreasing.

If $\{a_n\}$ is an increasing sequence, then $\{a_n\}$ is bounded below by a_1 and hence $\{a_n\}$ is bounded if and only if $\{a_n\}$ is bounded above. A decreasing sequence $\{a_n\}$ is bounded above by a_1 and hence $\{a_n\}$ is bounded if and only if $\{a_n\}$ is bounded below.

Monotone Subsequence and Convergence Theorem

Monotone Subsequence Theorem

If $\{a_n\}$ is a sequence of real numbers, then there is a monotone subsequence.

Monotone Convergence Theorem

A monotone sequence $\{a_n\}$ is convergent if and only if $\{a_n\}$ is bounded.

Bolzano-Weierstrass Theorem

Every bounded real sequence has a convergent subsequence.

(Proof)

From the Monotone Subsequence Theorem, if a real sequence is bounded, it has a monotone subsequence. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem that the subsequence is converge.

Cauchy Sequence

Definition: Cauchy Sequence

If $\{a_n\}$ is a sequence such that for every $\epsilon > 0$, there exists a positive integer N such that if $m, n \geq N$, we have

$$|a_m - a_n| < \epsilon$$

then we call $\{a_n\}$ a Cauchy sequence

The condition is known as the Cauchy condition.

Theorem

Let $\{a_n\}$ be a real sequence. Then $\{a_n\}$ is convergent if and only if $\{a_n\}$ is a Cauchy sequence.

Accumulation points

Definition

Let X be a subset of real numbers and let $a \in \mathbb{R}$. We say that a is an accumulation point of X if for every $\delta > 0$, there exists a number $x \in X$ such that $0 < |x - a| < \delta$.

In other words, a is an accumulation point of X if there exist points in X different from a which are arbitrarily near a. The number a may or may not belong to X.

a is a left accumulation point of X if for every $\delta > 0$, there exists a number $x \in X$ such that $0 < a - x < \delta$.
a is a right accumulation point of X if for every $\delta > 0$, there exists a number $x \in X$ such that $0 < x - a < \delta$.

Limit of a function

Definition

Let f be a function from a subset X of $\mathbb{R}$ into $\mathbb{R}$, and let a be an accumulation point of X. We say that the limit of a function $f(x)$ as x approaches a is L if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)-L| < \epsilon$ whenever $0 < |x-a| < \delta$. We write $\lim_{x\to a} f(x) = L$.

(Example) Consider $f(x) = 3x - 1$.
(In Calculus I) $\lim_{x\to 1} f(x) = \lim_{x\to 1}(3x - 1) = 2$.

In Analysis, we must prove this rigorously using the above definition!

We can show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

Proof

Show $\lim_{x \to 1} (3x - 1) = 2$.

$$\Downarrow$$

Find δ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

(Proof)

Let $\epsilon > 0$.

Suppose $0 < |x - 1| < \delta$.

Then $|(3x - 1) - 2| = |3(x - 1)| = 3|x - 1| < 3\delta$.

If we set $\delta = \frac{\epsilon}{3}$, $|(3x - 1) - 2| < \epsilon$ is satisfied.

Definition

Let f be a function from a subset X of $\mathbb{R}$ into $\mathbb{R}$ and let $a \in X$. Then f is continuous at a if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ and $x \in X$, then $|f(x) - f(a)| < \epsilon$.

When it is possible to calculate the limit of the function at a by simply substituting a for x, we say that the function is continuous at a.

Lemma

If f is continuous at c, there exists $\delta > 0$ such that f is bounded on $(c - \delta, c + \delta)$.

Heine-Borel Theorem

Let $\mathfrak{L}$ be a collection of open intervals such that

$$\cup \mathfrak{L} \supset [a, b] \, .$$

Then there exists a finite subset $\{I_1, ..., I_n\}$ of $\mathfrak{L}$ such that

$$\bigcup_{i=1}^{n} I_i \supset [a, b] \, .$$

The property of $[a, b]$ in this theorem is called "compactness".

Heine-Borel Theorem

Theorem

If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

Heine-Borel Theorem (In another way)

For a subset S of $\mathbb{R}$, the following two statements are equivalent:

1. S is closed and bounded.
2. Every open cover of S has a finite subcover, that is, S is compact.