

Analysis II: Basic knowledge of real analysis: Part II, Functions

Kenichi Maruno

Department of Mathematics, The University of Texas - Pan American

February 17, 2011

Accumulation points

Definition

Let X be a subset of real numbers and let $a \in \mathbb{R}$. We say that a is an accumulation point of X if for every $\delta > 0$, there exists a number $x \in X$ such that $0 < |x - a| < \delta$.

In other words, a is an accumulation point of X if there exist points in X different from a which are arbitrarily near a . The number a may or may not belong to X .

a is a left accumulation point of X if for every $\delta > 0$, there exists a number $x \in X$ such that $0 < a - x < \delta$.

a is a right accumulation point of X if for every $\delta > 0$, there exists a number $x \in X$ such that $0 < x - a < \delta$.

Limit of a function

Definition

Let f be a function from a subset X of \mathbb{R} into \mathbb{R} , and let a be an accumulation point of X . We say that the limit of a function $f(x)$ as x approaches a is L if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. We write $\lim_{x \rightarrow a} f(x) = L$.

(Example) Consider $f(x) = 3x - 1$.

(In Calculus I) $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (3x - 1) = 2$.

In Analysis, we must prove this rigorously using the above definition!

We can show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

Proof

Show $\lim_{x \rightarrow 1} (3x - 1) = 2$.



Find δ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

(Proof)

Let $\epsilon > 0$.

Suppose $0 < |x - 1| < \delta$.

Then $|(3x - 1) - 2| = |3(x - 1)| = 3|x - 1| < 3\delta$.

If we set $\delta = \frac{\epsilon}{3}$, $|(3x - 1) - 2| < \epsilon$ is satisfied.

Theorem

Let $f : X \rightarrow \mathbb{R}$, $X \subset \mathbb{R}$ and suppose a is an accumulation point of X . Then $\lim_{x \rightarrow a} f(x) = L$ if and only if for every sequence $\{a_n\}$ in X such that $\lim_{n \rightarrow \infty} a_n = a$ and $a_n \neq a$ for every positive integer n , we have $\lim_{n \rightarrow \infty} f(a_n) = L$

Continuity

Definition

Let f be a function from a subset X of \mathbb{R} into \mathbb{R} and let $a \in X$. Then f is continuous at a if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ and $x \in X$, then $|f(x) - f(a)| < \epsilon$.

When it is possible to calculate the limit of the function at a by simply substituting a for x , we say that the function is continuous at a .

Continuity: An Example

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show f is continuous at 0.

(Solution) Let $\epsilon > 0$. Then

$$\left| x \sin \left(\frac{1}{x} \right) - 0 \right| = \left| x \sin \left(\frac{1}{x} \right) \right| \leq |x|.$$

Choosing $\delta = \epsilon$, we see that

$$\left| x \sin \left(\frac{1}{x} \right) - 0 \right| \leq |x| < \epsilon,$$

for all $x \neq 0$ such that $|x - 0| < \delta$.

Uniform Continuity

Definition

Let f be a function on a closed interval $[a, b]$. Then f is uniformly continuous if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ for all $x, y \in [a, b]$, then $|f(x) - f(y)| < \epsilon$.

Theorem

Let f be a continuous function on the closed interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.

Heine-Borel Theorem

Lemma

If f is continuous at c , there exists $\delta > 0$ such that f is bounded on $(c - \delta, c + \delta)$.

Heine-Borel Theorem

Let \mathcal{L} be a collection of open intervals such that

$$\cup \mathcal{L} \supset [a, b].$$

Then there exists a finite subset $\{I_1, \dots, I_n\}$ of \mathcal{L} such that

$$\bigcup_{i=1}^n I_i \supset [a, b].$$

The property of $[a, b]$ in this theorem is called “compactness”.

Heine-Borel Theorem

Heine-Borel Theorem (In another way)

For a subset S of \mathbb{R} , the following two statements are equivalent:

1. S is closed and bounded.
2. Every open cover of S has a finite subcover, that is, S is compact.

Open cover means a possibly infinite collection of open intervals which include elements of S .

Using the Heine-Borel theorem, we can prove the following theorem.

Theorem

If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.