Analysis II: Riemann Integral

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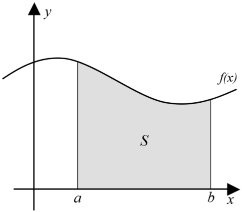


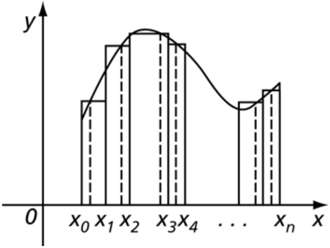
$$S = \int_a^b f(x)dx$$

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

If an antiderivative of $f(x)$ does not exist, we cannot use this formula.



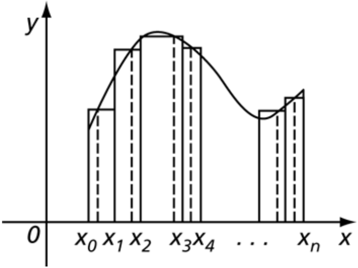
Definition: Partition

A partition P of a closed interval $[a, b]$ is a finite sequence $(x_0, x_1, \cdots, x_n)$ such that $a = x_0 < x_1 < \ldots < x_n = b$. The norm (or mesh) of P, denoted $||P||$, is defined by

$$||P|| = \max_{1 \le i \le n} (x_i - x_{i-1}) .$$

That is, $||P||$ is the length of the longest of the subintervals $[x_0, x_1]$, $[x_2, x_3]$,...$[x_{n-1}, x_n]$.

Riemann Integral



Definition: Riemann Sum

Let $P = (x_0, ..., x_n)$ be a partition of $[a, b]$, and let f be defined on $[a, b]$. For each $i = 1, ..., n$, let x_i^* be an arbitrary point in the interval $[x_{i-1}, x_i]$. Then any sum of the form

$$R(f, P) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is called a Riemann sum of f relative to P.

Definition: Upper Sum, Lower Sum

Suppose that f is bounded both above and below in each of the open intervals (x_{i-1}, x_i). Let $M_i = \sup_{(x_{i-1},x_i)} f$ and $m_i = \inf_{(x_{i-1},x_i)} f$. The upper sum of f relative to the partition $P = (x_0, \ldots, x_n)$ of $[a, b]$ is

$$\mathcal{U}(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \,.$$

The lower sum of f relative to the partition $P = (x_0, \ldots, x_n)$ of $[a, b]$ is

$$\mathcal{L}(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \,.$$

Note that

$$\mathcal{L}(f, P) \leq \mathcal{U}(f, P) \,.$$

Let $M = \sup_{[a,b]} f$ and $m = \inf_{[a,b]} f$. Since $M_i \le M$ for all i,

$$\begin{aligned} \mathcal{U}(f,P) &\le \sum_{i=1}^{n} M(x_i - x_{i-1}) \\ &= M\sum_{i=1}^{n}(x_i - x_{i-1}) = M(b-a)\,. \end{aligned}$$

Since $m_i \ge m$ for all i,

$$\begin{aligned} \mathcal{L}(f,P) &\ge \sum_{i=1}^{n} m(x_i - x_{i-1}) \\ &= m\sum_{i=1}^{n}(x_i - x_{i-1}) = m(b-a)\,. \end{aligned}$$

$$m(b-a) \le \mathcal{L}(f,P) \le \mathcal{U}(f,P)\,.$$

So $\mathcal{U}(f,P)$ is bounded below by $m(b-a)$.
Thus we can define the upper integral of f over $[a,b]$ by

$$\overline{\int_a^b} f = \inf \mathcal{U}(f,P)\,.$$

$$\mathcal{L}(f,P) \le \mathcal{U}(f,P) \le M(b-a)\,.$$

So $\mathcal{L}(f,P)$ is bounded above by $M(b-a)$.
Thus we can define the lower integral of f over $[a,b]$ by

$$\underline{\int_a^b} f = \sup \mathcal{L}(f,P)\,.$$

Theorem

Let f be a bounded function on the interval $[a, b]$. Then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f \,.$$

Definition:Riemann Integrable

f is Riemann integrable over $[a, b]$ if

$$\overline{\int_a^b} f = \underline{\int_a^b} f .$$

We denote the value by

$$\int_a^b f$$

or

$$\int_a^b f(x)dx$$

Definition: Riemann Integrable

A function f is Riemann integrable on $[a, b]$ if there is a real number R such that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any partition P of $[a, b]$ satisfying $||P|| < \delta$, and for any Riemann sum $R(f, P)$ of f relative to P, we have $|R(f, P) - R| < \epsilon$.

$$\lim_{||P|| \to 0} R(f, p) = \int_a^b f \quad \left(\text{or} \quad \int_a^b f(x)dx \right)$$

Show that the function $f(x) = x$ is Riemann integrable in $[0, 1]$, and that

$$\int_0^1 f = \frac{1}{2}\,.$$

(Solution)
Let P be the partition $(0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1)$. In the subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ $(i = 1, 2, ..., n)$,

$$M_i = \sup\left[\frac{i-1}{n}, \frac{i}{n}\right] = \frac{i}{n}\,,$$

$$m_i = \inf\left[\frac{i-1}{n}, \frac{i}{n}\right] = \frac{i-1}{n}\,.$$

So

$$\mathcal{U}(f,P)=\sum_{i=1}^{n}\frac{i}{n}\frac{1}{n}=\frac{1}{n^2}(1+2+\cdots+n)=\frac{1}{n^2}\frac{n(n+1)}{2}=\frac{1}{2}+\frac{1}{2n}\,.$$

and

$$\mathcal{L}(f,P)=\sum_{i=1}^{n}\frac{i-1}{n}\frac{1}{n}=\frac{1}{n^2}(0+1+\cdots+(n-1))=\frac{1}{n^2}\frac{(n-1)n}{2}=\frac{1}{2}-\frac{1}{2n}\,.$$

Since

$$\overline{\int_0^1} f\le\mathcal{U}(f,P)\,,\quad \underline{\int_0^1} f\ge\mathcal{L}(f,P)\,,\quad \underline{\int_0^1} f\le\overline{\int_0^1} f\,,$$

the inequality

$$0\le\overline{\int_0^1} f-\underline{\int_0^1} f\le\mathcal{U}(f,P)-\mathcal{L}(f,P)=\frac{1}{n}$$

holds for every n.

$$\overline{\int_0^1} f = \underline{\int_0^1} f$$

as $n \to \infty$. So f is Riemann integrable. Since we know

$$\frac{1}{2} - \frac{1}{2n} \leq \int_0^1 f \leq \frac{1}{2} + \frac{1}{2n}$$

for every $n \geq 1$, we conclude that

$$\int_0^1 f = \frac{1}{2}\,.$$

The characteristic function and the Dirichlet function

Definition: The characteristic function

With any set of real numbers A, we associate a function χ_A, called the characteristic function of A, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Definition: The Dirichlet function

Let Q be the set of rational numbers.

$$\chi_Q(x) = \begin{cases} 1 & \text{if} \quad x \in Q \\ 0 & \text{if} \quad x \notin Q \end{cases}$$

is called the Dirichlet function.

The Dirichlet function

The Dirichlet function $\chi_Q(x)$ on $[a,b]$ is not Riemann integrable.
(Proof)
For every partition $P=\{x_0(=a),x_1,...,x_{n-1},x_n(=b)\}$, every subinterval (x_{i-1},x_i) contains both rational and irrational points. So, for $i=1,2,...,n$,

$$M_i=1,\quad m_i=0\,.$$

Hence, for every partition P,

$$\mathcal{U}(f,P)=b-a\,,\quad \mathcal{L}(f,P)=0$$

So

$$\overline{\int_a^b} f=b-a\,,\quad \underline{\int_a^b} f=0$$

Thus f is not Riemann integrable.

The Dirichlet function

The Dirichlet function $\chi_Q(x)$ is not Riemann integrable.

The Dirichlet function is obtained as a limit:

$$\chi_Q(x) = \lim_{n\to\infty}\left[\lim_{m\to\infty}(\cos n!\pi x)^{2m}\right]$$

Functions similar to the Dirichlet function often appear in engineering problems.

But these functions are not Riemann integrable!

This is a big problem for applications.

$\Downarrow$

Lebesgue introduced another definition of integral: Lebesgue integral

The Dirichlet function is Lebesgue intagrable!

Riemann Integral

Definition: Step function

A function g, defined on $[a, b]$, is a step function if there is a partition $P = (x_0, x_1, ..., x_n)$ such that g is constant on each open subinterval (x_{i-1}, x_i), for $i = 1, ..., n$.

Proposition

Any step function g on $[a, b]$ is Riemann integrable. Furthermore, if $g(x) = c_i$ for $x \in (x_{i-1}, x_i)$, where $(x_0, x_1, ..., x_n)$ is a partition of $[a, b]$, then

$$\int_a^b g(x)dx = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) \,.$$

Riemann Integral

Theorem

A function f, defined om $[a, b]$, is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there are step functions f_1 and f_2 such that

$$f_1(x) \leq f(x) \leq f_2(x) \quad \text{for all} \quad x \in [a, b]\,,$$

and

$$\int_a^b f_2(x)dx - \int_a^b f_1(x)dx < \epsilon\,.$$

Corollary

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \text{lub}\{\int_a^b f_1(x)dx | f_1 \quad \text{a step function and} f_1 \leq f\}$$

$$= \text{glb}\{\int_a^b f_2(x)dx | f_2 \quad \text{a step function and} f \leq f_2\}$$