# Analysis II: Fourier Series

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# Fourier Series

Fourier series were introduced by Joseph Fourier (17681830) for the purpose of solving the heat equation in a metal plate. The heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}\,.$$

Jean Baptiste Joseph Fourier (21 March 1768  16 May 1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. Fourier is also generally credited with the discovery of the greenhouse effect.

# Fourier Series

To solve a problem in the heat equation, Fourier needed to express a function $f$ as an infinite series of sine and cosine functions:

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &\quad + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \end{aligned}$$

This is called a trigonometric series or Fourier series.

Expressing a function as a Fourier series is sometimes more advantageous than expanding it as a power series.

Applications: astronomical phenomena, heartbeats, tides, vibrating strings, ocean waves, sound waves, music, etc.(periodic phenomena)

Let $f(x)$ be a continuous function on $[-\pi, \pi]$. Then assume that we can express $f(x)$ by the trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \,, \qquad -\pi \le x \le \pi$$

We must determine $a_n$ and $b_n$!
Integrate the above expression:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)dx$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \, dx \,.$$

## Fourier Series

$$\int_{-\pi}^{\pi} \cos nx\, dx = \frac{1}{n} \sin nx\Big]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0\,,$$

and

$$\int_{-\pi}^{\pi} \sin nx\, dx = -\frac{1}{n} \cos nx\Big]_{-\pi}^{\pi} = -\frac{1}{n}[\cos n\pi - \cos(-n\pi)] = 0\,.$$

So

$$\int_{-\pi}^{\pi} f(x) dx = 2\pi a_0\,.$$

Thus

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

To determine $a_n$ for $n \geq 1$, we multiply both sides of the equation by $\cos mx$ where $m$ is an integer and $m \geq 1$:

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos mx\,dx \\
&\quad = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx\,dx \\
&= a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx \\
&\qquad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx
\end{aligned}$$

$$\int_{-\pi}^{\pi} \cos mx\, dx = 0 \text{ for all } m$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0 \text{ for all } n \text{ and } m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \begin{cases} 0 \text{ for } n \neq m \\ \pi \text{ for } n = m \end{cases}$$

(Orthogonality)
So

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = a_m \pi\,.$$

Thus

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx\,.$$

To determine $a_n$ for $n \geq 1$, we multiply both sides of the equation by $\sin mx$ where $m$ is an integer and $m \geq 1$:

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x) \sin mx \, dx \\
&\quad = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \, dx \\
&= a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx \\
&\qquad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx
\end{aligned}$$

$$\int_{-\pi}^{\pi} \sin mx\, dx = 0 \text{ for all } m$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx\, dx = 0 \text{ for all } n \text{ and } m$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \begin{cases} 0 \text{ for } n \neq m \\ \pi \text{ for } n = m \end{cases}$$

(Orthogonality)
So

$$\int_{-\pi}^{\pi} f(x) \sin mx\, dx = b_m \pi\,.$$

Thus

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\, dx\,.$$

Definition: Fourier Series

Let $f$ be a piecewise continuous function on $[-\pi, \pi]$. Then the Fourier series of $f$ is the series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients $a_n$ and $b_n$ in this series are defined by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \,,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \,,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \,.$$

and are called the Fourier coefficients of $f$.

Euler formula

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta\,.$$

So

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2}\,, \quad \sin\theta = \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{2\mathrm{i}}\,.$$

So

$$e^{n\mathrm{i}x} = \cos nx + \mathrm{i}\sin nx\,.$$

So

$$\cos nx = \frac{e^{\mathrm{i}nx} + e^{-\mathrm{i}nx}}{2}\,, \quad \sin nx = \frac{e^{\mathrm{i}nx} - e^{-\mathrm{i}nx}}{2\mathrm{i}}\,.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{e^{inx}+e^{-inx}}{2}\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{i}\frac{-e^{inx}+e^{-inx}}{2}\,dx$$
$$= -\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx + \frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,,$$

So

$$\frac{a_n+ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx\,,\quad \frac{a_n-ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,.$$

$$
\begin{aligned}
& a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\
&= a_0 + \sum_{n=1}^{\infty} \left( a_n \cdot \frac{e^{inx} + e^{-inx}}{2} + b_n \cdot i \frac{-e^{inx} + e^{-inx}}{2} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \left( \frac{a_n - ib_n}{2} \cdot e^{inx} + \frac{a_n + ib_n}{2} \cdot e^{-inx} \right) \\
&= c_0 + \sum_{n=1}^{\infty} \left( c_n \, e^{inx} + c_{-n} \, e^{-inx} \right) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}
\end{aligned}
$$

where

$$
c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \, ,
$$

$$
c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx \, ,
$$

$$
c_0 = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \, ,
$$

# Fourier Series

**Definition: Complex Form of Fourier Series**

Let $f$ be a piecewise continuous function on $[-\pi, \pi]$. Then the Fourier series of $f$ is the series

$$\sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

where the Fourier coefficients $c_n$ in this series are defined by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \,,$$

for all integers $n$.

# Fourier Series

We can generalize Fourier series for functions of period $L$.

**Definition: Complex Form of Fourier Series**

Let $f$ be a piecewise continuous function on $[-L/2, L/2]$. Then the Fourier series of $f$ is the series

$$\sum_{n=-\infty}^{\infty} c_n \, e^{\frac{2\pi}{L} inx}$$

where the Fourier coefficients $c_n$ in this series are defined by

$$c_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-\frac{2\pi}{L} inx} \, dx \, ,$$

for all integers $n$.

# Fourier Transform

The Fourier transform is a generalization of the complex Fourier series in the limit as $L \to \infty$.

Fourier transform of $f(x)$:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx$$

The Fourier transform has many applications. Any field of physical science that uses sinusoidal signals, such as engineering, physics, applied mathematics, and chemistry, will make use of Fourier series and Fourier transforms.