Definite Integral

\[ S = \int_{a}^{b} f(x) \, dx \]

The Fundamental Theorem of Calculus

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

where \( F(x) \) is an antiderivative of \( f(x) \).

If an antiderivative of \( f(x) \) does not exist, we cannot use this formula.
Riemann Integral

A partition \( P \) of a closed interval \([a, b]\) is a finite sequence \((x_0, x_1, \cdots, x_n)\) such that \( a = x_0 < x_1 < \cdots < x_n = b \). The norm (or mesh) of \( P \), denoted \( ||P|| \), is defined by

\[
||P|| = \max_{1 \leq i \leq n} (x_i - x_{i-1}).
\]

That is, \( ||P|| \) is the length of the longest of the subintervals \([x_0, x_1], [x_2, x_3], \ldots, [x_{n-1}, x_n] \).
Definition: Riemann Sum

Let \( P = (x_0, \ldots, x_n) \) be a partition of \([a, b]\), and let \( f \) be defined on \([a, b]\). For each \( i = 1, \ldots, n \), let \( x_i^* \) be an arbitrary point in the interval \([x_{i-1}, x_i]\). Then any sum of the form

\[
R(f, P) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})
\]

is called a Riemann sum of \( f \) relative to \( P \).
Suppose that $f$ is bounded both above and below in each of the open intervals $(x_{i-1}, x_i)$. Let $M_i = \sup_{(x_{i-1}, x_i)} f$ and $m_i = \inf_{(x_{i-1}, x_i)} f$. The upper sum of $f$ relative to the partition $P = (x_0, ..., x_n)$ of $[a, b]$ is

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}).$$

The lower sum of $f$ relative to the partition $P = (x_0, ..., x_n)$ of $[a, b]$ is

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}).$$

Note that

$$L(f, P) \leq U(f, P).$$
Let $M = \sup_{[a,b]} f$ and $m = \inf_{[a,b]} f$. Since $M_i \leq M$ for all $i$,

$$U(f, P) \leq \sum_{i=1}^{n} M(x_i - x_{i-1})$$

$$= M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(b - a) .$$

Since $m_i \geq m$ for all $i$,

$$L(f, P) \geq \sum_{i=1}^{n} m(x_i - x_{i-1})$$

$$= m \sum_{i=1}^{n} (x_i - x_{i-1}) = m(b - a) .$$
\[ m(b - a) \leq \mathcal{L}(f, P) \leq \mathcal{U}(f, P). \]

So \( \mathcal{U}(f, P) \) is bounded below by \( m(b - a) \).

Thus we can define the upper integral of \( f \) over \([a, b]\) by

\[
\int_a^b f = \inf \mathcal{U}(f, P).
\]

\[ \mathcal{L}(f, P) \leq \mathcal{U}(f, P) \leq M(b - a). \]

So \( \mathcal{L}(f, P) \) is bounded above by \( M(b - a) \).

Thus we can define the lower integral of \( f \) over \([a, b]\) by

\[
\int_a^b f = \sup \mathcal{L}(f, P).
\]
Theorem
Let $f$ be a bounded function on the interval $[a, b]$. Then

$$\int_{a}^{b} f \leq \int_{a}^{b} f.$$
Definition: Riemann Integrable

A function $f$ is Riemann integrable over $[a, b]$ if

$$
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f.
$$

We denote the value by

$$
\int_{a}^{b} f
$$

or

$$
\int_{a}^{b} f(x) \, dx
$$
Definition: Riemann Integrable

A function \( f \) is Riemann integrable on \([a, b]\) if there is a real number \( R \) such that for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for any partition \( P \) of \([a, b]\) satisfying \( \|P\| < \delta \), and for any Riemann sum \( R(f, P) \) of \( f \) relative to \( P \), we have \( |R(f, P) - R| < \epsilon \).

\[
\lim_{\|P\| \to 0} R(f, p) = \int_{a}^{b} f \quad \text{(or} \quad \int_{a}^{b} f(x) \, dx)\]

K.Maruno (UT-Pan American)
Show that the function \( f(x) = x \) is Riemann integrable in \([0, 1]\), and that
\[
\int_0^1 f = \frac{1}{2}.
\]

(Solution)
Let \( P \) be the partition \((0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)\). In the subinterval \([\frac{i-1}{n}, \frac{i}{n}]\) \((i = 1, 2, \ldots, n)\),
\[
M_i = \sup \left[ \frac{i-1}{n}, \frac{i}{n} \right] = \frac{i}{n},
\]
\[
m_i = \inf \left[ \frac{i-1}{n}, \frac{i}{n} \right] = \frac{i-1}{n}.
\]
So

\[ U(f, P) = \sum_{i=1}^{n} \frac{i}{n} \frac{1}{n} = \frac{1}{n^2} (1 + 2 + \cdots + n) = \frac{1}{n^2} \frac{n(n + 1)}{2} = \frac{1}{2} + \frac{1}{2n}. \]

and

\[ L(f, P) = \sum_{i=1}^{n} \frac{i - 1}{n} \frac{1}{n} = \frac{1}{n^2} (0 + 1 + \cdots + (n - 1)) = \frac{1}{n^2} \frac{(n - 1)n}{2} = \frac{1}{2} - \frac{1}{2n}. \]

Since

\[ \int_{0}^{1} f \leq U(f, P), \quad \int_{0}^{1} f \geq L(f, P), \quad \int_{0}^{1} f \leq \int_{0}^{1} f, \]

the inequality

\[ 0 \leq \int_{0}^{1} f - \int_{0}^{1} f \leq U(f, P) - L(f, P) = \frac{1}{n} \]

holds for every \( n \).
\[ \int_{0}^{1} f = \int_{0}^{1} f \]
as \( n \to \infty \). So \( f \) is Riemann integrable. Since we know

\[ \frac{1}{2} - \frac{1}{2n} \leq \int_{0}^{1} f \leq \frac{1}{2} + \frac{1}{2n} \]

for every \( n \geq 1 \), we conclude that

\[ \int_{0}^{1} f = \frac{1}{2}. \]
Definition: The characteristic function

With any set of real numbers $A$, we associate a function $\chi_A$, called the characteristic function of $A$, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition: The Dirichlet function

Let $Q$ be the set of rational numbers.

$$\chi_Q(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

is called the Dirichlet function.
The Dirichlet function $\chi_Q(x)$ on $[a, b]$ is not Riemann integrable.

(Proof)

For every partition $P = \{x_0 (= a), x_1, \ldots, x_{n-1}, x_n (= b)\}$, every subinterval $(x_{i-1}, x_i)$ contains both rational and irrational points. So, for $i = 1, 2, \ldots, n$,

$$M_i = 1, \quad m_i = 0.$$ 

Hence, for every partition $P$,

$$\mathcal{U}(f, P) = b - a, \quad \mathcal{L}(f, P) = 0$$

So

$$\int_a^b f = b - a, \quad \int_{-a}^b f = 0$$

Thus $f$ is not Riemann integrable.
The Dirichlet function $\chi_Q(x)$ is not Riemann integrable. The Dirichlet function is obtained as a limit:

$$\chi_Q(x) = \lim_{n \to \infty} \left[ \lim_{m \to \infty} (\cos n! \pi x)^{2m} \right]$$

Functions similar to the Dirichlet function often appear in engineering problems. But these functions are not Riemann integrable! This is a big problem for applications.

Lebesgue introduced another definition of integral: Lebesgue integral. The Dirichlet function is Lebesgue integrable!
Definition: Step function

A function $g$, defined on $[a, b]$, is a step function if there is a partition $P = (x_0, x_1, \ldots, x_n)$ such that $g$ is constant on each open subinterval $(x_{i-1}, x_i)$, for $i = 1, \ldots, n$.

Proposition

Any step function $g$ on $[a, b]$ is Riemann integrable. Furthermore, if $g(x) = c_i$ for $x \in (x_{i-1}, x_i)$, where $(x_0, x_1, \ldots, x_n)$ is a partition of $[a, b]$, then

$$
\int_a^b g(x) \, dx = \sum_{i=1}^n c_i (x_i - x_{i-1}).
$$
**Theorem**

A function $f$, defined on $[a, b]$, is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there are step functions $f_1$ and $f_2$ such that

$$f_1(x) \leq f(x) \leq f_2(x) \quad \text{for all} \quad x \in [a, b],$$

and

$$\int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx < \epsilon.$$

**Corollary**

If $f$ is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \operatorname{lub}\{\int_a^b f_1(x)\,dx \mid f_1 \text{ a step function and } f_1 \leq f\}$$

$$= \operatorname{glb}\{\int_a^b f_2(x)\,dx \mid f_2 \text{ a step function and } f \leq f_2\}$$